

EIGENVALUES OF VECTOR FIELDS, BOTT'S RESIDUE FORMULA AND INTEGRAL INVARIANTS

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ABSTRACT. Given a compatible vector field on a compact connected almost-complex manifold, we show in this article that the multiplicities of eigenvalues among the zero point set of this vector field have intimate relations. We highlight a special case of our result and reinterpret it as a vanishing-type result in the framework of the celebrated Atiyah-Bott-Singer localization formula. This new point of view, via the Chern-Weil theory and a strengthened version of Bott's residue formula observed by Futaki and Morita, can lead to an obstruction to Killing real holomorphic vector fields on compact Hermitian manifolds in terms of a curvature integral.

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1. INTRODUCTION

In an earlier article [11], the author showed that, if a compact connected almost-complex manifold admits a compatible circle action with nonempty fixed points, the weights among the fixed point set of this action have intimate relations [11, Theorem 1.1]. The main idea of the proof in [11] is refined from a beautiful observation of Lusztig in [12], which is the invariance of the equivariant Hirzebruch χ_y -genus under compact connected Lie group actions. Recently, the author also notices that Lusztig's this consideration is closely related to Futaki and Morita's work on the reinterpretation of the Futaki integral invariant on Fano manifolds ([6], [7], [5, Chapter 5]). In fact some considerations of Futaki and Morita could be improved

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by using the observation in [12] and some other results, which has been clarified by the author in [10].

The main purpose of the present article is twofold. On the one hand, we shall show that the idea developed in [11] can be carried over to the case of vector fields on compact almost-complex manifolds to yield a similar result (Theorem 2.2). On the other hand, we highlight a special case of this result (Corollary 2.3) by rephrasing it as a vanishing-type result in the framework of the Atiyah-Bott-Singer localization formula. This new point of view, through the Chern-Weil theory and a strengthened version of Bott's residue formula, can lead to a vanishing result of curvature integral when the underlying manifolds are complex and the vector fields are Killing (Theorem 2.6).

Here it is worth pointing out that the idea of this application has a new feature, which is converse to the usual philosophy of localization methods involving vector fields. The latter is to localize the investigation of a global property of the manifolds to the consideration of local information around the zero points of vector fields. While our method is to show a property related to some local information around the zero points of vector fields and piece it up into a global one.

The rest of this article is arranged as follows. In Section 2 we will state our main results in this article, Theorems 2.2 and 2.6 and Corollaries 2.3 and 2.8. We recall the Atiyah-Bott-Singer localization formula in Section 3 and reinterpret our Corollary 2.3 as a vanishing-type result in this framework. In Section 4 we firstly introduce a strengthened version of Bott's residue formula, which includes the invariant polynomials whose degrees are larger than the dimension of the underlying manifold and has been observed by Futaki and Morita to reinterpret the famous Futaki invariant. Then combining this version of Bott's residue formula with our reinterpretation of Corollary 2.3 leads to the proof of Theorem 2.6. Although the main idea of the proof of Theorem 2.2 is similar to that as in [11], for completeness and the reader's convenience, we still in Section 5 present its proof.

2. STATEMENT OF MAIN RESULTS

Suppose (M^n, J) is a compact connected almost-complex manifold with complex dimension n and a fixed almost-complex structure J . A smooth vector field A on (M^n, J) is called *compatible* if it preserves the almost-complex structure J and the one parameter group of A , $\exp(tA)$, lies in a *compact* group. The latter condition is equivalent to the condition that the vector field A preserves an almost-Hermitian metric on (M^n, J) , i.e., A is *Killing* with respect to this almost-Hermitian metric.

Given this (M^n, J) and a compatible vector field A whose zero point set is *nonempty*, we choose an almost-Hermitian metric g such that A is Killing with respect to this g . Let $\text{zero}(A)$ denote the zero point set of A . As is well-known in the compact transformation group theory ([8]), $\text{zero}(A)$ consists of finitely many connected components and each one is a compact almost-Hermitian submanifold in M . Moreover, the normal bundle of each connected component in $\text{zero}(A)$ can be splitted into a sum of complex line bundles with respect to the skew-Hermitian transformation induced by A . Let Z be any such a connected component with complex dimension r . Here r of course depends on the choice of Z in $\text{zero}(A)$. Then the normal bundle of Z in M , denoted $\nu(Z)$, can be decomposed into a sum of $n - r$ complex line

bundles

$$\nu(Z) = \bigoplus_{i=1}^{n-r} L(Z, \lambda_i), \quad \lambda_i \in \mathbb{R} - \{0\},$$

such that the eigenvalue of the skew-Hermitian transformation induced by A on the line bundle $L(Z, \lambda_i)$ is $\sqrt{-1}\lambda_i$. Or equivalently, the eigenvalue of the action induced by the one-parameter group $\exp(tA)$ on $L(Z, \lambda_i)$ is $\exp(\sqrt{-1}\lambda_i t)$. Note that these nonzero real numbers $\lambda_1, \dots, \lambda_{n-r}$ are counted with multiplicities and thus not necessarily mutually distinct. Of course they depend on the choice of Z in $\text{zero}(A)$. Note also that these λ_i are actually independent of the almost-Hermitian metric g we choose and completely determined by the vector field A .

From now on we use “ $e(\cdot)$ ” to denote the Euler characteristic of a manifold.

Definition 2.1. Let us attach a set $S(A)$ to the vector field A as follows.

$$S(Z) := \begin{cases} \coprod_{e(Z) \text{ copies}} \{\lambda_1, \dots, \lambda_{n-r}\}, & \text{if } e(Z) > 0, \\ \emptyset, & \text{if } e(Z) = 0, \\ \coprod_{-e(Z) \text{ copies}} \{-\lambda_1, \dots, -\lambda_{n-r}\}, & \text{if } e(Z) < 0. \end{cases}$$

and

$$S(A) := \coprod_Z S(Z),$$

where the sum is over all the connected components in $\text{zero}(A)$. Here the symbol “ \coprod ” means *disjoint union*. That means, although we write $\{\lambda_1, \dots, \lambda_{n-r}\}$ as a set, repeated elements in it may *not* be discarded.

Now we can state our main result in this section, which shows that the eigenvalues on the connected components of $\text{zero}(A)$ whose Euler numbers are nonzero have intimate relations.

Theorem 2.2. Let the notation be as above. Then for any $\lambda \in S(A)$, the multiplicity of λ in $S(A)$ is the same as that of $-\lambda$ in $S(A)$.

By our Definition 2.1, Theorem 2.2 itself provides no information on those connected components whose Euler numbers are zero. However, the following direct corollary, which will be rephrased as a vanishing-type result in terms of the Atiyah-Bott-Singer localization formula in the next section and play an important role in our applications, can include them.

Corollary 2.3. With the above notation understood, we have

$$\sum_Z [e(Z) \cdot \sum_{i=1}^{n-r} \lambda_i] = 0,$$

the sum being over all the connected components in $\text{zero}(A)$.

Remark 2.4.

- (1) When $\text{zero}(A)$ consists of isolated zero points, various special cases of Theorem 2.2 and Corollary 2.3 have been obtained in previous literature by using different methods ([13, Theorem 2], [9, Theorem 3.5], [6, Corollary 6.3], [5, Corollary 5.3.12]).
- (2) When $\text{zero}(A)$ is arbitrary and the compatible vector field A generates a circle action, it has been obtained by the author in [11].

The following typical example illustrates Theorem 2.2 very well.

Example 2.5. Let $\mathbb{C}P^n$ be the n -dimensional complex projective space with homogeneous coordinate $[z_0, z_1, \dots, z_n]$ and $\lambda_1, \dots, \lambda_s$ ($s \leq n+1$) be s mutually distinct real numbers. We arbitrarily choose s nonnegative integers n_1, \dots, n_s such that

$$\sum_{i=1}^s (n_i + 1) = n + 1.$$

Using these data we can define a one-parameter group action ψ_t on $\mathbb{C}P^n$ by

$$\begin{aligned} \psi_t : \mathbb{C}P^n &\longrightarrow \mathbb{C}P^n, \\ [z_0, z_1, \dots, z_n] &\longmapsto \\ [e^{\sqrt{-1}\lambda_1 t} z_0, \dots, e^{\sqrt{-1}\lambda_1 t} z_{n_1}, e^{\sqrt{-1}\lambda_2 t} z_{n_1+1}, \dots, e^{\sqrt{-1}\lambda_2 t} z_{n_1+n_2+1}, \dots, e^{\sqrt{-1}\lambda_s t} z_{n-n_s}, \dots, e^{\sqrt{-1}\lambda_s t} z_n], \end{aligned}$$

i.e., each λ_i appears exactly $n_i + 1$ times consecutively. Let A be the vector field generating this ψ_t . Thus

$$\text{zero}(A) = \text{fixed point set of the action } \{\psi_t\} = \prod_{i=1}^s M_i,$$

where

$$M_i = \{[0, \dots, 0, z_{k_i}, \dots, z_{k_i+n_i}, 0, \dots, 0] \in \mathbb{C}P^n\} \cong \mathbb{C}P^{n_i},$$

and

$$k_i = \sum_{j=1}^{i-1} (n_j + 1) \quad (i \geq 2), \quad k_1 := 0, \quad \mathbb{C}P^0 := \{pt\}.$$

The eigenvalues of the vector field A on the connected component M_i are

$$\{\sqrt{-1}(\lambda_j - \lambda_i) \text{ with multiplicity } n_j + 1 \mid j \neq i\}.$$

Thus in $S(A)$, the multiplicity of each $\lambda_j - \lambda_i$ ($1 \leq i, j \leq s, i \neq j$) is $(n_i + 1)(n_j + 1)$ as $e(M_i) = n_i + 1$.

To distinguish the symbols (M^n, J) and A for an almost-complex manifold and its compatible vector field, we use once and for all $(N, J) = (N^n, J)$ and X to denote a compact complex manifold of complex dimension n and a *real holomorphic* vector field on N respectively. Here by “real holomorphic” we mean that the corresponding vector field $X - \sqrt{-1}JX$ on $T^{1,0}N$, the $(1,0)$ -part of the complexified tangent bundle of N , is holomorphic.

Before stating our next result, we need to introduce some notation and symbols.

We choose a Hermitian metric g on (N, J) and let ∇ be the Hermitian connection on the holomorphic tangent bundle $T^{1,0}N$, which is also called in some literature the *Chern connection*. Let $\Gamma(\text{End}(T^{1,0}N))$ be the vector space of smooth sections of the endomorphism bundle of $T^{1,0}N$. Following Bott ([2, Lemma 1]), we define an element $L(X) \in \Gamma(\text{End}(T^{1,0}N))$ as follows. Put

$$L(X)(\cdot) := [X, \cdot] - \nabla_X(\cdot) : \Gamma(T^{1,0}N) \rightarrow \Gamma(T^{1,0}N),$$

where $[\cdot, \cdot]$ is the Lie bracket and $\Gamma(T^{1,0}N)$ is the vector space of smooth sections of $T^{1,0}N$. The stability of $\Gamma(T^{1,0}N)$ under the map $L(X)$ has been explained in [2, Lemma 1]. Moreover, for any smooth function f and $Y \in \Gamma(T^{1,0}N)$, it is direct to verify that $L(X)(fY) = fL(X)(Y)$ by using the derivation property of ∇ . Thus $L(X)$ can be viewed as

an $\text{End}(T^{1,0}N)$ -valued function, i.e., $L(X) \in \Gamma(\text{End}(T^{1,0}N))$. Let R be the curvature form of ∇ , which is an $\text{End}(T^{1,0}N)$ -valued $(1, 1)$ -form on N , i.e., $R \in \Gamma(\text{End}(T^{1,0}N) \otimes T^{1,0}N \otimes \overline{T^{1,0}N})$. Therefore it makes sense to discuss the trace $(\text{tr}(\cdot))$ and determinant $(\det(\cdot))$ of $L(X)$ and R (see Theorem 2.6 below).

After reinterpreting Corollary 2.3 as a vanishing-type result in terms of the Atiyah-Bott-Singer residue formula in Section 3 and building a bridge via the Chern-Weil theory in Section 4, we can yield the following result, which provides an obstruction-type result to the holomorphic Killing vector fields on compact complex manifolds.

Theorem 2.6. Suppose (N^n, J, g) is an n -dimensional compact Hermitian manifold and X is a Killing and real holomorphic vector field. Then we have

$$\int_M \text{tr}(L(X)) \cdot c_n(\nabla) + \int_M c_1(\nabla) \cdot \det\left(L(X) + \frac{\sqrt{-1}}{2\pi}R\right) = 0.$$

Here $c_i(\nabla)$ is the i -th Chern form with respect to the Chern connection ∇ .

Remark 2.7. Note that $\det\left(L(X) + \frac{\sqrt{-1}}{2\pi}R\right)$ is a differential form of possibly mixed degrees. So in this theorem we are only concerned with its homogeneous component of $(n-1, n-1)$ -form due to the dimensional reason.

Theorem 2.6 has the following corollary when the Hermitian metric g is Kähler.

Corollary 2.8. If (N^n, J, g) is an n -dimensional compact Kähler manifold and X is a Killing vector field, we have

$$\int_M \text{Ric}(g) \cdot \det\left(\nabla X + \frac{\sqrt{-1}}{2\pi}R\right) = 0,$$

where $\text{Ric}(g)$ is the Ricci form of the Kähler metric g .

Proof. If the Hermitian metric g is Kähler, then the Chern connection ∇ coincides with the Levi-Civita connection of g , which implies

$$L(X)(\cdot) = [X, \cdot] - \nabla_X(\cdot) = -\nabla_{(\cdot)}X = -\nabla X.$$

On the other hand, a well-known fact ([8, p. 107, Theorem 4.3]) tells us that a smooth vector field X on a compact Kähler manifold is Killing if and only if it is real holomorphic and its divergence $\text{div}(X) = 0$. By definition we have $\text{div}(X) = \text{tr}(\nabla X)$ and thus $\text{tr}(L(X)) = -\text{tr}(\nabla X) = 0$ if X is Killing. Also note that in this case the first Chern form $c_1(\nabla) = \text{Ric}(g)$. Therefore,

$$\begin{aligned} 0 &= \int_M c_1(\nabla) \cdot \det\left(L(X) + \frac{\sqrt{-1}}{2\pi}R\right) \\ &= \int_M \text{Ric}(g) \cdot \det\left(-\nabla(X) + \frac{\sqrt{-1}}{2\pi}R\right) \\ &= -\int_M \text{Ric}(g) \cdot \det\left(\nabla(X) + \frac{\sqrt{-1}}{2\pi}R\right). \end{aligned}$$

□

3. ATIYAH-BOTT-SINGER LOCALIZATION FORMULA AND REINTERPRETATION OF COROLLARY 2.3

As before let (M^n, J) be a compact connected almost-complex manifold and A its compatible vector field with nonempty zero point set $\text{zero}(A)$. We keep using the related notation and symbols introduced in Section 2.

In this section, we briefly recall the Atiyah-Bott-Singer residue formula, which reduces the calculation of the Chern numbers of (M^{2n}, J) to the consideration of the local information around $\text{zero}(A)$.

Let x_1, \dots, x_n denote the formal Chern roots of M , i.e., the total Chern class of (M, J) , $c(M, J)$, has the following formal decomposition:

$$c(M, J) := 1 + \sum_{i=1}^n t^i \cdot c_i(M, J) = \prod_{i=1}^n (1 + tx_i).$$

Similarly, we denote by $\alpha_1, \dots, \alpha_r$ the formal Chern roots of the connected component Z in $\text{zero}(A)$ and $\beta_1, \dots, \beta_{n-r}$ the Euler classes (or the first Chern classes) of the complex line bundles $L(Z, \lambda_1), \dots, L(Z, \lambda_{n-r})$.

With the above-defined notation and symbols in mind, we have the following localization formula, which reduces the calculation of the Chern numbers of (M^{2n}, J) to the consideration of the local information around $\text{zero}(A)$ [1, p. 598].

Theorem 3.1 (Residue formula, almost-complex case). Let $\varphi = \varphi(\cdot, \dots, \cdot)$ be a symmetric polynomial with n variables and we define

$$f_\varphi(A) := \sum_Z \int_Z \frac{\varphi(\alpha_1, \dots, \alpha_r, \sqrt{-1}\lambda_1 + \beta_1, \dots, \sqrt{-1}\lambda_{n-r} + \beta_{n-r})}{\prod_{j=1}^{n-r} (\sqrt{-1}\lambda_j + \beta_j)}.$$

If we use $\deg(\varphi)$ to denote the degree of the symmetric polynomial φ . Then

$$f_\varphi(A) = \begin{cases} 0, & \deg(\varphi) < n \\ \int_M \varphi(x_1, \dots, x_n), & \deg(\varphi) = n. \end{cases}$$

Remark 3.2. When J is integrable, i.e., (M, J) is a compact complex manifold, Theorem 3.1 was established by Bott in [2, Theorem 1] ($\text{zero}(A)$ is isolated) and [3, Theorem 2] (general case) by using direct differential-geometric arguments. The current version was established by Atiyah and Singer in [1, § 8], which is a beautiful application of their general Lefschetz fixed point formula.

Note that, even if $\deg(\varphi) > n$, $f_\varphi(A)$ is still *well-defined*. But Theorem 3.1 says *nothing* for those φ whose degrees are *larger* than n . Our first observation in this section is that our Corollary 2.3 is equivalent to a vanishing result of $f_\varphi(A)$ for some φ whose degree is $n+1$. To be more precise, if we use $c_i = c_i(\cdot, \dots, \cdot)$ ($1 \leq i \leq n$) to denote the i -th elementary symmetric polynomial with n variables, Corollary 2.3 can be rephrased as follows.

Proposition 3.3. $f_{c_1 c_n}(A) \equiv 0$ for any compatible vector field A on (M^{2n}, J) .

Proof.

$$\begin{aligned}
& f_{c_1 c_n}(A) \\
&= \sum_Z \int_Z \frac{c_1(\cdots) c_n(\cdots)}{\prod_j^{n-r} (\sqrt{-1} \lambda_j + \beta_j)} \\
&= \sum_Z \int_Z \frac{[\sum_{i=1}^r \alpha_i + \sum_{i=1}^{n-r} (\sqrt{-1} \lambda_i + \beta_i)] \cdot [\prod_i^r \alpha_i \cdot \prod_i^{n-r} (\sqrt{-1} \lambda_i + \beta_i)]}{\prod_j^{n-r} (\sqrt{-1} \lambda_j + \beta_j)} \\
&= \sum_Z \int_Z \left\{ \left[\sum_{i=1}^r \alpha_i + \sum_{i=1}^{n-r} (\sqrt{-1} \lambda_i + \beta_i) \right] \cdot \prod_{i=1}^r \alpha_i \right\} \\
&= \sum_Z [e(Z) \cdot \sum_{j=1}^{n-r} \sqrt{-1} \lambda_j] \\
&= 0.
\end{aligned}$$

The fourth equality is due to the fact that $\prod_{i=1}^r \alpha_i$ is nothing but the Euler class of Z and $\dim_{\mathbb{C}} Z = r$. \square

Remark 3.4.

- (1) When M is a Fano manifold, i.e., a compact Kähler manifold with positive first Chern class, Futaki and Morita showed that ([6], [7]), up to some constant, $f_{c_1^{n+1}}(A)$ is nothing but the Futaki integral invariant with respect to the holomorphic vector field A . This gives a geometric interpretation of $f_{\varphi}(A)$ for $\varphi = c_1^{n+1}$ when M is Kähler. In contrast with this, it is somewhat surprising to see that $f_{c_1 c_n}(A) \equiv 0$ for any compatible vector field A on any compact almost-complex manifold M .
- (2) When M is Kähler, A is nondegenerate and $\text{zero}(A)$ only consists of isolated fixed points. Proposition 3.3 has been obtained by Futaki and Morita ([6], [5, p. 80]). It is their this observation, together with their reinterpretation of the Futaki integral invariant on Fano manifolds, that inspire our this proposition and the current article.

As we have mentioned, Theorem 3.1 says nothing for those φ whose degrees are larger than n . However, Futaki and Morita noticed that ([6], [7], [4, Chapter 5]), for compact complex manifolds, Bott's original arguments in [2] and [3] can also be applied to including $\{\varphi \mid \deg(\varphi) > n\}$. As an application, they showed that, the famous Futaki integral invariant, which was introduced by Futaki in [4] and obstructs the existence of Kähler-Einstein metrics on Fano manifolds, can be put into this framework as a special case.

Now let us give a precise statement of Bott residue formula in this strengthened version.

We denote by $I^k(gl(n, \mathbb{C}))$ ($0 \leq k \leq n$) the $GL(n, \mathbb{C})$ -invariant polynomial function of degree k on $gl(n, \mathbb{C})$, i.e., $\varphi \in I^k(gl(n, \mathbb{C}))$ means that

$$\begin{aligned}
& \varphi : gl(n, \mathbb{C}) \rightarrow \mathbb{C}, \\
& \varphi((a_{ij})_{n \times n}) = \sum \lambda_{i_1 \cdots i_k j_1 \cdots j_k} a_{i_1 j_1} \cdots a_{i_k j_k},
\end{aligned}$$

where

$$(a_{ij})_{n \times n} \in gl(n, \mathbb{C}) \quad \text{and} \quad \lambda_{i_1 \cdots i_k j_1 \cdots j_k} \in \mathbb{C},$$

and satisfies

$$\varphi(PAP^{-1}) = \varphi(A), \quad \forall A \in gl(n, \mathbb{C}), \quad \forall P \in GL(n, \mathbb{C}).$$

It is well-known that

$$I^*(gl(n, \mathbb{C})) := \bigoplus_{k \geq 0} I^k(gl(n, \mathbb{C}))$$

is multiplicatively generated by c_i ($0 \leq i \leq n$), which are characterized by

$$\det(I_n + tA) =: \sum_{i=0}^n t^i \cdot c_i(A), \quad I_n = n \times n \text{ identity matrix.}$$

Remark 3.5. By a slight abuse of symbols, c_i has at least four different meanings in our article: the i -th elementary symmetric polynomial, the i -th generator of $GL(n, \mathbb{C})$ -invariant polynomial, the i -th Chern form $c_i(\nabla) := c_i(\frac{\sqrt{-1}}{2\pi}R)$, and the i -th Chern class of a complex vector bundle. The reason for this abuse is clear to those who are familiar with the Chern-Weil theory.

Given $\varphi \in I^*(gl(n, \mathbb{C}))$, with the above symbols and notation understood, we know that $\varphi(L(X) + \frac{\sqrt{-1}}{2\pi}R)$ is a well-defined differential form of possibly mixed degrees on N . The following beautiful residue formula of Bott tells us that, under some reasonable requirement on X , i.e., X is *non-degenerate* (see Remark 3.7 for a precise definition), the evaluation of $\varphi(L(X) + \frac{\sqrt{-1}}{2\pi}R)$ on N can be localized to that of $\text{zero}(X)$.

Theorem 3.6 (Bott's residue formula, complex case). With the above materials understood and assume that X is non-degenerate. For any $\varphi \in I^*(gl(n, \mathbb{C}))$, we have

$$\begin{aligned} f_\varphi(X) &:= \int_M \varphi(L(X) + \frac{\sqrt{-1}}{2\pi}R) \\ (3.1) \quad &= \sum_{Z \subset \text{zero}(X)} \int_Z \frac{\varphi(L(X)|_Z + \frac{\sqrt{-1}}{2\pi}R|_Z)}{\det(L^\mu(X) + \frac{\sqrt{-1}}{2\pi}R^\mu)}, \end{aligned}$$

where the sum is over all the connected components Z of $\text{zero}(X)$, $(\cdot)|_Z$ denotes the restriction to Z , $L^\mu(X) \in \Gamma(\text{End}(\mu(Z)))$ is the induced section of the normal bundle $\mu(Z)$ from $L(X)$, and R^μ is the curvature form of $\mu(Z)$ with respect to the induced Hermitian metric.

Some more remarks related to Theorem 3.6 are in order.

Remark 3.7.

- (1) The precise meaning of *non-degenerate* is that ([3, p. 314]) each Z is a complex submanifold of N and the kernel of the endomorphism $L(X)|_Z$ on $T^{1,0}N|_Z$ is precisely $T^{1,0}Z$. Thus non-degeneracy guarantees that both the denominator and the integral on the right hand side of (3.1) be well-defined. When this X is Killing, which is the requirement in Theorem 3.1, it is always non-degenerate according to the *compact* transformation group theory. So despite the integrability condition, the assumption on the vector field in this theorem is weaker than that in Theorem 3.1. The reason is that the proof of the latter is based on the Lefschetz fixed point formula of Aityah-Bott-Segal-Singer, which needs the group acted on the manifold to be compact.

- (2) When $\deg(\varphi) \leq n$, we have

$$\int_M \varphi(L(X) + \frac{\sqrt{-1}}{2\pi}R) = \int_M \varphi(\frac{\sqrt{-1}}{2\pi}R)$$

and thus (3.1) becomes

$$\int_M \varphi\left(\frac{\sqrt{-1}}{2\pi}R\right) = \sum_{Z \subset \text{zero}(X)} \int_Z \frac{\varphi(L(X)|_Z + \frac{\sqrt{-1}}{2\pi}R|_Z)}{\det(L^\mu(X) + \frac{\sqrt{-1}}{2\pi}R^\mu)}.$$

This, via the Chern-Weil theory, exactly corresponds to the localization formula in Theorem 3.1. This is what Bott's original statement presents. Using the current version of Bott's residue formula, Futaki-Morita showed that, up to some constant factor, the original Futaki invariant is essentially equal to $f_{c_1^{n+1}}(X)$ and so the calculation of the Futaki invariant can also be localized to the zero point of the holomorphic vector field explicitly by Theorem 3.6. Later, Bott's idea was further extracted by Tian in [14, § 6] to give a residue formula of the Calabi-Futaki integral invariant, which obstructs the existence of constant scalar curvature metrics in a given Kähler class.

- (3) A detailed proof of (3.1) can be found in [5, Theorem 5.2.8]. However, an implicit but more concise proof can also be found in [15, p. 313].

By virtue of the above discussion, combining Proposition 3.3 with Theorem 3.6 can immediately yield the following result.

Proposition 3.8. Suppose (N^{2n}, J, g) is a compact Hermitian manifold, X a Killing and real holomorphic vector field on N and ∇ the Chern connection. Then we have

$$\int_M c_1 c_n \left(L(X) + \frac{\sqrt{-1}}{2\pi} R \right) = \sum_Z [\text{tr}(L^\mu(X)) \cdot e(Z)] = 0.$$

4. APPLICATIONS

In this section, we will present two applications of Propositions 3.3 and 3.8. The first one is the proof of Theorem 2.6, which is almost immediate from Proposition 3.8. The second one is based on an interesting observation in [6].

First, we prove Theorem 2.6, which we restate here again.

Theorem 4.1. Suppose (N^{2n}, J, g) is a compact Hermitian manifold and X is a Killing and real holomorphic vector field. Then we have

$$\int_M \text{tr}(L(X)) \cdot c_n(\nabla) + \int_M c_1(\nabla) \cdot \det\left(L(X) + \frac{\sqrt{-1}}{2\pi}R\right) = 0.$$

Here $c_i(\nabla) := c_i(\frac{\sqrt{-1}}{2\pi}R)$ is the i -th Chern form with respect to the Chern connection ∇ .

Proof. By Proposition 3.8 we have

$$\int_M c_1 c_n \left(L(X) + \frac{\sqrt{-1}}{2\pi} R \right) = 0.$$

Note that

$$\begin{aligned}
& c_1 c_n \left(L(X) + \frac{\sqrt{-1}}{2\pi} R \right) \\
&= \text{tr} \left(L(X) + \frac{\sqrt{-1}}{2\pi} R \right) \cdot \det \left(L(X) + \frac{\sqrt{-1}}{2\pi} R \right) \\
&= \left[\text{tr}(L(X)) + \text{tr} \left(\frac{\sqrt{-1}}{2\pi} R \right) \right] \\
&\quad \cdot \left[\det \left(\frac{\sqrt{-1}}{2\pi} R \right) + \{ \det(L(X) + \frac{\sqrt{-1}}{2\pi} R) \}^{(n-1)} + \text{lower degree terms} \right] \\
&= \left[\text{tr}(L(X)) + c_1(\nabla) \right] \cdot \left[c_n(\nabla) + \{ \det(L(X) + \frac{\sqrt{-1}}{2\pi} R) \}^{(n-1)} + \text{lower degree terms} \right] \\
&= \text{tr}(L(X)) \cdot c_n(\nabla) + c_1(\nabla) \cdot \{ \det(L(X) + \frac{\sqrt{-1}}{2\pi} R) \}^{(n-1)}.
\end{aligned}$$

Here $\{\dots\}^{(n-1)}$ means the component of $(n-1, n-1)$ -form in $\{\dots\}$. Now the equality in the above theorem follows easily from this deduction. \square

Remark 4.2. It is clear that the statement itself is purely differential-geometric. But the author does not know whether or not this result could be proved by a direct differential-geometric argument.

Our second application in this section is based on an observation in [6], which has been mentioned in Remark 3.4.

Besides the reinterpretation of the Futaki invariant, Futaki and Morita also proved many other interesting results related to $f_\varphi(X)$. [6] is an announcement of the properties of these integral invariants, whose proofs are contained in [7] and chapter 5 of Futaki's book [5]. For example, they found that $c_1 c_n$ is the *unique* monomial of degree $n+1$ satisfying $f_{c_1 c_n}(X) \equiv 0$ for *any* non-degenerate holomorphic vector field whose zero points are *isolated* on *any* Kähler manifold. Now combining their observation and our Proposition 3.3, we have the following vanishing-type and uniqueness result.

Theorem 4.3. Among all the monomials of degree $n+1$

$$\{c_1^{s_1} \cdots c_n^{s_n} \mid \sum_{i=0}^n i \cdot s_i = n+1, s_i \in \mathbb{Z}, s_i \geq 0\},$$

$c_1 c_n$ is the *unique* monomial which satisfies $f_{c_1 c_n}(A) = 0$ for *any* compatible vector field A on *any* almost-complex manifold (M^{2n}, J) .

5. PROOF OF THEOREM 2.2

The proof of Theorem 2.2 is an interesting application of the rigidity property of the Hirzebruch χ_y -genus. This rigidity phenomenon, which was first observed by Lusztig in [12], is a striking application of the general Lefschetz fixed point formula developed by Atiyah, Bott, Segal and Singer. Recently the author refines this observation and gives some applications to symplectic geometry and related topics ([9], [11]).

The idea of the proof of Theorem 2.2 basically follows that of the main theorem in [11]. However, for the reader's convenience, we still sketch its proof.

As usual we use $\bar{\partial}$ to denote the d -bar operator which acts on the complex vector spaces $\Omega^{p,q}(M)$ ($0 \leq p, q \leq n$) of (p, q) -type differential forms on (M^{2n}, J) in the sense of J . The choice of the almost Hermitian metric g on (M^{2n}, J) enables us to define the Hodge star operator $*$ and the formal adjoint $\bar{\partial}^* = - * \bar{\partial} *$ of the $\bar{\partial}$ -operator. Then for each $0 \leq p \leq n$, we have the following Dolbeault-type elliptic operator

$$(5.1) \quad \bigoplus_{q \text{ even}} \Omega^{p,q}(M) \xrightarrow{\bar{\partial} + \bar{\partial}^*} \bigoplus_{q \text{ odd}} \Omega^{p,q}(M),$$

whose index is denoted by $\chi^p(M)$ in the notation of Hirzebruch. We define the Hirzebruch χ_y -genus, $\chi_y(M)$, by

$$\chi_y(M) := \sum_{p=0}^n \chi^p(M) \cdot y^p.$$

The general form of the Hirzebruch-Riemann-Roch theorem allows us to compute $\chi_y(M)$ in terms of the Chern numbers of M as follows.

$$\chi_y(M) = \int_M \prod_{i=1}^n \frac{x_i(1 + ye^{-x_i})}{1 - e^{-x_i}}.$$

The proof of Theorem 2.2 can be divided into the following four steps, which we encode by four lemmas.

STEP 1.

The first lemma we present below is refined from [12], which is a beautiful application of the Lefschetz fixed point theorem for elliptic complexes (5.1).

Lemma 5.1. *The following identity holds*

$$(5.2) \quad \chi_y(M) \equiv \sum_Z \int_Z \left(\prod_{i=1}^r \alpha_i \frac{1 + ye^{-\alpha_i}}{1 - e^{-\alpha_i}} \right) \left(\prod_{j=1}^{n-r} \frac{1 + ye^{\sqrt{-1}\lambda_j t} e^{-\beta_j}}{1 - e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}} \right), \quad \forall t,$$

i.e., the right-hand side of (5.2), when taken as a rational function of t , is identically equal to $\chi_y(M)$.

Proof. The action of the one-parameter group $\exp(tA)$ on (M, J) can be lifted to the elliptic complex (5.1). Thus we can define the equivariant index $\chi^p(t, M)$ and the equivariant χ_y -genus $\chi_y(t, M) := \sum_{p=0}^n \chi^p(t, M) \cdot y^p$. The Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer ([1, p. 562]) allows us to compute $\chi_y(t, M)$ in terms of the local information around the fixed point set of the one-parameter group action $\exp(tA)$, which is exactly $\text{zero}(A)$, as follows.

$$(5.3) \quad \chi_y(t, M) = \sum_Z \int_Z \left(\prod_{i=1}^r \alpha_i \frac{1 + ye^{-\alpha_i}}{1 - e^{-\alpha_i}} \right) \left(\prod_{j=1}^{n-r} \frac{1 + ye^{\sqrt{-1}\lambda_j t} e^{-\beta_j}}{1 - e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}} \right).$$

Note that the right-hand side of (5.3) has well-defined limits as $\sqrt{-1}t$ tends to $+\infty$ and $-\infty$:

$$\lim_{\sqrt{-1}t \rightarrow +\infty} (\text{RHS of (5.3)}) = \sum_Z \chi_y(Z) (-y)^{d_+(Z)},$$

$$\lim_{\sqrt{-1}t \rightarrow -\infty} (\text{RHS of (5.3)}) = \sum_Z \chi_y(Z) (-y)^{d_-(Z)},$$

where $d_+(Z)$ (resp. $d_-(Z)$) is the number of positive (resp. negative) numbers among the eigenvalues $\lambda_1, \dots, \lambda_{n-r}$. So the left-hand side of (5.3) and thus each $\chi^p(t, M)$ also have well-defined limits as $\sqrt{-1}t$ tends to $+\infty$ and $-\infty$.

But by definition, for each $0 \leq p \leq n$, $\chi^p(t, M)$ is the trace of the action $\exp(tA)$ on the complex representation space $\ker(\bar{\partial} + \bar{\partial}^*) - \text{coker}(\bar{\partial} + \bar{\partial}^*)$. Note that A is Killing with respect to the metric g . Thus $\chi^p(t, M)$ is of the following *finite* sum

$$\chi^p(t, M) = \sum_i a_i(p) \cdot \exp(\sqrt{-1}t\theta_i(p)), \quad a_i(p) \in \mathbb{Z} - \{0\}, \theta_i(p) \in \mathbb{R}.$$

So the only possibility that $\chi^p(t, M)$ has well-defined limits as $\sqrt{-1}t$ tends to $+\infty$ and $-\infty$ is $\theta_i(p) \equiv 0$ and therefore $\chi^p(t, M)$ is a constant for any t . This completes the proof of (5.2) and Step 1. \square

STEP 2.

In this step we calculate a coefficient in the Taylor expansion of (5.2) at $y = -1$. The motivation that inspires us to investigate the coefficient of $y + 1$ has been explained in details in [11, Section 3], which comes from another interesting phenomenon of the Hirzebruch χ_y -genus.

Lemma 5.2. *Note that the right-hand side of (5.2) is a polynomial of y . If we consider its Taylor expansion at $y = -1$, the coefficient of the first order term $y + 1$ is*

$$(5.4) \quad \sum_Z \left[\left(\frac{r}{2} - n \right) e(Z) + e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right]$$

Proof. From [11, Lemma 2.3] we know that

$$\prod_{i=1}^r \frac{\alpha_i(1 + ye^{-\alpha_i})}{1 - e^{-\alpha_i}} = c_r(Z) + [c_{r-1}(Z) - \frac{r}{2}c_r(Z)] \cdot (y + 1) + \dots,$$

and

$$\begin{aligned}
\prod_{j=1}^{n-r} \frac{1 + ye^{\sqrt{-1}\lambda_j t} e^{-\beta_j}}{1 - e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}} &= \prod_{j=1}^{n-r} \left[1 + \frac{e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}}{1 - e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}} (y + 1) \right] \\
&= 1 + \left(\sum_{j=1}^{n-r} \frac{e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}}{1 - e^{\sqrt{-1}\lambda_j t} e^{-\beta_j}} \right) (y + 1) + \cdots \\
&= 1 + \left(\sum_{j=1}^{n-r} \frac{e^{\sqrt{-1}\lambda_j t}}{1 - e^{\sqrt{-1}\lambda_j t}} + \text{higher degree terms} \right) \cdot (y + 1) + \cdots.
\end{aligned}$$

Combining these two expressions we can obtain that the coefficient of $y+1$ on the right-hand side of (5.2) is

$$\begin{aligned}
&\sum_Z \left[-\frac{r}{2}e(Z) + e(Z) \sum_{j=1}^{n-r} \frac{e^{\sqrt{-1}\lambda_j t}}{1 - e^{\sqrt{-1}\lambda_j t}} \right] \\
&= \sum_Z \left[-\frac{r}{2}e(Z) + e(Z) \sum_{j=1}^{n-r} \left(-1 + \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right) \right] \\
&= \sum_Z \left[\left(\frac{r}{2} - n \right) e(Z) + e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right],
\end{aligned}$$

which is precisely (5.4). This completes Step 2. \square

STEP 3.

We carefully compare the coefficients of the term $y + 1$ on both sides of (5.2) and obtain the following result.

Lemma 5.3.

$$\begin{aligned}
(5.5) \quad &\sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) > 0}} \left[e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}t\lambda_j}} \right] + \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) < 0}} \left[-e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{-\sqrt{-1}t\lambda_j}} \right] \\
&\equiv \frac{1}{2} \sum_{Z \subset \text{zero}(A)} (n-r)|e(Z)|, \quad \forall t.
\end{aligned}$$

Here “ $|\cdot|$ ” means taking the absolute value.

Proof. The coefficient of the left-hand side of (5.2) is $-\frac{n}{2}e(M)$ ([11, Lemma 2.3]), which, together with (5.4), yields

$$(5.6) \quad -\frac{n}{2}e(M) \equiv \sum_Z \left[\left(\frac{r}{2} - n \right) e(Z) + e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right], \quad \forall t.$$

Note that we have the famous result $e(M) = \sum_Z e(Z)$, which, for example, can be obtained by taking $y = -1$ in (5.2) as $\chi_y(\cdot)|_{y=-1} = e(\cdot)$.

Using this identity $e(M) = \sum_Z e(Z)$ to substitute $e(M)$ in (5.6), we obtain

$$\begin{aligned}
& \frac{1}{2} \sum_Z (n-r) e(Z) \\
& \equiv \sum_Z \left[e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right] \\
(5.7) \quad & = \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) > 0}} \left[e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right] + \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) < 0}} \left[-e(Z) \sum_{j=1}^{n-r} \frac{-1}{1 - e^{\sqrt{-1}\lambda_j t}} \right] \\
& = \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) > 0}} \left[e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} \right] + \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) < 0}} \left[-e(Z) \sum_{j=1}^{n-r} \left(\frac{1}{1 - e^{\sqrt{-1}\lambda_j t}} - 1 \right) \right], \quad \forall t.
\end{aligned}$$

Rewriting (5.7) slightly, we can easily yield (5.5). □

STEP 4.

We can now complete the proof of Theorem 2.2.

Lemma 5.4. *Theorem 2.2 holds.*

Proof. Set $h := \exp(\sqrt{-1}t)$. Then (5.5) becomes

$$\begin{aligned}
(5.8) \quad & \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) > 0}} \left[e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - h^{\lambda_j}} \right] + \sum_{\substack{Z \subset \text{zero}(A) \\ e(Z) < 0}} \left[-e(Z) \sum_{j=1}^{n-r} \frac{1}{1 - h^{-\lambda_j}} \right] \\
& \equiv \frac{1}{2} \sum_{Z \subset \text{zero}(A)} (n-r) |e(Z)|, \quad \forall h.
\end{aligned}$$

Now Theorem 2.2 follows from (5.8), [11, Lemma 2.4] and [11, Remark 2.5]. □

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